PARABOLA

1. The locus of a point which moves in a plane so that its distance from a fixed point bears a constant ratio to its distance from a fixed straight line is called a ***conic section*** or ***conic***. The fixed point is called ***focus,*** the fixed straight line is called ***directrix*** and the constant ratio 'e' is called ***eccentricity*** of the conic.

2. If e = 1, then the conic is called a ***parabola.***

3. If e < 1, then the conic is called an ***ellipse.***

4. If e > 1, then the conic is called a ***hyperbola.***

5. The equation of a conic is of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

6. A line L = 0 passing through the focus of a conic is said to be the ***principal axis*** of the conic if it is perpendicular to the directrix of the conic.

7. The points of intersection of a conic and its principal axis are called ***vertices*** of the conic.

8. If a conic has only one vertex then its centre coincides with the vertex.

9. A conic has at most two vertices.

10. The midpoint of the line segment joining the vertices of a conic is called ***centre*** of the conic.

11. A conic is said to be in the ***standard form*** if the principal axis of the conic is x-axis and the centre of the conic is the origin.

12. The equation of a parabola in the standard form is $y^2 = 4ax$.

13. For the parabola $y^2 = 4ax$, vertex=(0, 0), focus=(a, 0) and the equation of the directrix is x+a=0.

14. If we rotate the axes 90° in the clockwise direction then the equation $y^2 = 4ax$ of a parabola is transformed to $x^2 = 4ay$.

15. For the parabola $x^2 = 4ay$, vertex = (0, 0), focus = (0, a), the equation of the directrix is y+a = 0 and the equation of the principal axis is x = 0 (y-axis).

16. A point (x_1, y_1) is said to be an

    i) ***external point*** of the parabola $y^2 = 4ax$ if $y_1^2 - 4ax_1 > 0$

    ii) ***internal point*** of the parabola $y^2 = 4ax$ if $y_1^2 - 4ax_1 < 0$.

17. A chord passing through a point P on the parabola and perpendicular to the principal axis of the parabola is called the ***double ordinate*** of the P.

18. A chord of the parabola passing through the focus is called a ***focal chord.***

19. A focal chord of a parabola perpendicular to the principal axis of the parabola is called ***latus rectum.*** If the latus rectum meets the parabola in L and L′ , then LL′ is called ***length of the latus rectum.***

20. The length of the latus rectum of the parabola $y^2 = 4ax$ is $4|a|$.

21. If P is a point on the parabola with focus S, then SP is called ***focal distance*** of P.

22. The focal distance of $P(x_1, y_1)$ on the parabola $y^2 = 4ax$ is $|x_1 + a|$.

23. The equation of the parabola whose axis is parallel to x-axis and vertex at (α, β) is $(y - \beta)^2 = \pm 4a(x - \alpha)$, $(a>0)$.

24. For the parabola $(y - \beta)^2 = \pm 4a(x - \alpha)$, the focus is $(\alpha \pm a, \beta)$ and the equation to the directrix is $x = \alpha \mp a$.

25. The equation $(y - \beta)^2 = \pm 4a(x - \alpha)$ can be put in the form $x = ly^2 + my + n$.

26. The equation of the parabola whose axis is parallel to y-axis and vertex at (α, β) is $(x - \alpha)^2 = \pm 4a(y - \beta)$

27. For the parabola $(x - \alpha)^2 = \pm 4a(y - \beta)$, the focus is $(\alpha, \beta \pm a)$, the equation of the directrix is $y = \beta \mp a$.

28. The equation $(x - \alpha)^2 = 4a(y - \beta)$ can be put in the form $y = lx^2 + mx + n$.

29. We use the following notation in this chapter

    $S \equiv y^2 - 4ax$, $S_1 \equiv yy_1 - 2a(x+x_1)$, $S_{11} = S(x_1, y_1) \equiv y_1^2 - 4ax_1$, $S_{12} \equiv y_1y_2 - 2a(x_1 + x_2)$.

30. Let $P(x_1, y_1)$ be a point and $S \equiv y^2 - 4ax = 0$ be a parabola. Then

    i) P lies on the parabola $\Leftrightarrow S_{11} = 0$

    ii) P Lies inside the parabola $\Leftrightarrow S_{11} < 0$

    iii) P lies outside the parabola $\Leftrightarrow S_{11} > 0$

31. The equation of the chord joining the two points $A(x_1, y_1)$, $B(x_2, y_2)$ on the parabola $S = 0$ is $S_1 + S_2 = S_{12}$.

32. Let S= 0 be a parabola and P be a point on the parabola. Let Q be any other point on the parabola. If the secant line $\overleftrightarrow{PQ}$ approaches to the same limiting position as Q moves along the curve and approaches to P form either side, then the limiting position is called a ***tangent line*** or ***tangent*** to the parabola at P. The point P is called ***point of contact*** of the tangent to the parabola.

33. If L = 0 is a tangent to the parabola S = 0 at P, then we say that the line L = 0 touches the parabola S= 0 at P.

34. The equation of the tangent to the parabola S = 0 at $P(x_1, y_1)$ is $S_1 = 0$.

35. Let S = 0 be a parabola and P be a point on the parabola S = 0. The line passing through P and perpendicular to the tangent of S = 0 at P is called the ***normal*** to the parabola S = 0 at P.

36. The equation of the normal to the parabola $y^2 = 4ax$ at $P(x_1, y_1)$ is $y_1(x - x_1) + 2a(y - y_1) = 0$.

37. The condition that the line $y = mx + c$ may be a tangent to the parabola $y^2 = 4ax$ is $c = a/m$.

38. The equation of a tangent to the parabola $y^2 = 4ax$ may be taken as $y = mx + a/m$. The point of contact is $(a/m^2, 2a/m)$.

39. If m_1, m_2 are the slopes of the tangents of the parabola $y^2 = 4ax$ through an external point $P(x_1, y_1)$, then $m_1 + m_2 = y_1/x_1$, $m_1m_2 = a/x_1$.

40. The line joining the points of contact of the tangent to a parabola S = 0 drawn from an external point P is called ***chord of contact*** of P with respect to the parabola S = 0.

41. The equation to the chord of contact of $P(x_1, y_1)$ with respect to the parabola S = 0 is $S_1 = 0$.

42. The locus of the point of intersection of the tangents to the parabola S = 0 brawn at the extremities of the chord passing through a point P is a straight line L = 0, called the ***polar*** of P with respect to the parabola S =0. The point P is called the ***pole*** of the line L = 0 with respect to the parabola S = 0.

43. The equation of the polar of the point $P(x_1, y_1)$ with respect to the parabola S = 0 is $S_1 = 0$.

44. If P is an external point of the parabola S = 0, then the polar of P meets the parabola in two points and the polar becomes the chord of contact of P.

45. If P lies on the parabola S = 0, then the polar of P becomes the tangent at P to the parabola S=0.

46. If P is an internal point of the parabola S = 0, then the polar of P does not meet the parabola.

47. The pole of the line $lx + my + n = 0$ ($l \neq 0$) with respect to the parabola $y^2 = 4ax$ is $(n/l, -2am/l)$.

48. Two points P and Q are said to be ***conjugate points*** with respect to the parabola S =0 if the polar of P with respect to S = 0 passes through Q.

49. The condition for the points $P(x_1, y_1)$, $Q(x_2, y_2)$ to be conjugate with respect to the parabola S = 0 is $S_{12} = 0$.

50. Two lines $L_1 = 0$, $L_2 = 0$ are said to be ***conjugate lines*** with respect to the parabola S = 0 if the pole of $L_1 = 0$ lie on $L_2 = 0$.

51. The condition for the lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ to be conjugate with respect to the parabola $y^2 = 4ax$ is $l_1n_2 + l_2n_1 = 2am_1m_2$.

52. The equation of the chord of the parabola S = 0 having $P(x_1, y_1)$ as its midpoint is $S_1 = S_{11}$.

53. The equation to the pair of tangents to the parabola S =0 from $P(x_1, y_1)$ is $S_1^2 = S_{11}S$.

54. A point (x, y) on the parabola $y^2 = 4ax$ can be represented as $x=at^2$, $y = 2at$ in a single parameter t. These equations are called ***parametric equations*** of the parabola $y^2 = 4ax$. The point $(at^2, 2at)$ is simply denoted by t.

55. The equation of the chord joining the points t_1 and t_2 on the parabola $y^2 = 4ax$ is $y(t_1 + t_2) = 2x + 2at_1t_2$.

56. If the chord joining the points t_1 and t_2 on the parabola $y^2 = 4ax$ is a focal chord then $t_1t_2 = -1$.

57. The equation of the tangent to the parabola $y^2 = 4ax$ at the point 't' is $yt = x + at^2$.

58. The point of intersection of the tangents to the parabola $y^2 = 4ax$ at the points t_1 and t_2 is $(at_1t_2, a[t_1+t_2])$.

59. The equation of the normal to the parabola $y^2 = 4ax$ at the point t is $y + xt = 2at + at^3$.

60. Three normals can be drawn from a point (x_1, y_1) to the parabola $y^2 = 4ax$.

61. If t_1, t_2, t_3 are the feet of the three normals drawn from point (x_1, y_1) to the parabola $y^2 = 4ax$ then $t_1 + t_2 + t_3 = 0$, $t_1t_2 + t_2t_3 + t_3t_1 = \frac{2a - x_1}{a}, t_1t_2t_3 = \frac{y_1}{a}$.

62. If the normals at t_1 and t_2 to the parabola meet on the parabola, then $t_1t_2 = 2$.

63. For the parabola $x = ly^2 + my + n$,
Vertex= $\left(n - \frac{m^2}{4l}, \frac{-m}{2l}\right)$, Focus = $\left(n + \frac{1-m^2}{4l}, \frac{-m}{2l}\right)$,

Latusrectum = $\frac{1}{l}$, axis is y+ $\frac{m}{2l}$=0, directrix is x=n+$\frac{1-m^2}{4l}$.

64. For the parabola $y = lx^2 + mx + n$,
Vertex=$\left(-\frac{m}{2l}, n - \frac{m^2}{4l}\right)$,Focus = $\left(\frac{-m}{2l}, n + \frac{1-m^2}{4l}\right)$,

Latusrectum = $\frac{1}{l}$, axis is y + $\frac{m}{2l}$=0, directrix is y=n+$\frac{1-m^2}{4l}$.

65. The condition that the line $lx + my + n = 0$ may be a tangent to the parabola

i) $y^2 = 4ax$ is $am^2 = ln$ ii) $x^2 = 4ay$ is $a\,l^2 = mn$.

66. The pole of the line $lx + my + n = 0$ $(m \neq 0)$ with respect to the parabola

i) $y^2 = 4ax$ is $\left(\frac{n}{l}, -\frac{2am}{l}\right)$ ii) $x^2 = 4ay$ is $\left(-\frac{2al}{m}, \frac{n}{m}\right)$

67. The length of the chord joining t_1, t_2 on $y^2 = 4ax$ is $a\,|\,t_1 - t_2\,|\,\sqrt{(t_1 + t_2)^2 + 4}$.

68. The length of the focal chord through the point t on the parabola $y^2 = 4ax$ is $a(t + 1/t)^2$.

69. If the normal at t_1 on the parabola $y^2 = 4ax$ meets it again at t_2 then $t_2 = -t_1 - 2/t_1$.

70. If the normal at t on the parabola $y^2 = 4ax$ subtends a right angle

i) at its focus then $t = \pm 2$ ii) at its vertex then $t = \pm\sqrt{2}$

71. The orthocentre of the triangle formed by three tangents of a parabola lies on the directrix

72. The angle between the pair of tangents drawn from (x_1, y_1) to the parabola $S \equiv y^2 - 4ax = 0$ is $\tan^{-1}\frac{\sqrt{S_{11}}}{x_1 + a}$

TABLES FORM OF CONIC SECTION : (FORMULAS)

PARABOLA								
S.No	Equation	Vertex	Focus	Latus rectum	Axis	Tangent at vertex	Directix	Equation of L.R.
i)	$y^2 = 4ax$	(0, 0)	(a, 0)	4 |a|	y = 0	x = 0	x + a = 0	x – a = 0
ii)	$(y-k)^2=4a(x-h)$	(h, k)	(a + h, k)	4 |a|	y – k=0	x– h=0	x– h +a=0	x–h–a = 0
iii)	$(x-h)^2=4a(y-k)$	(h, k)	(h, k + a)	4 |a|	x – h=0	y– k=0	y–k + a=0	y–k–a = 0

Horizontal ellipse (a > b) e = $\frac{\sqrt{a^2-b^2}}{a}$ or $\sqrt{1-\frac{b^2}{a^2}}$								
Equation	Centre	Focii	Directricies	Major axis	Minor axis	Latus rectum	Verticies	Property
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	(0, 0)	$(\pm\sqrt{a^2-b^2},0)$	$(\sqrt{a^2-b^2})x = \pm a^2$	y = 0	x = 0	$2\frac{b^2}{a}$	(±a, 0) (0, ±b)	$SP+S^1P = 2a$
Vertical ellipse (a > b) e = $\sqrt{1-\frac{a^2}{b^2}}$								
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	(0, 0)	$(0,\pm\sqrt{b^2-a^2})$	$(\sqrt{b^2-a^2})y = \pm b^2$	x = 0	y = 0	$2\frac{a^2}{b}$	(± a, 0) (0, ± b)	$SP+S^1P = 2b$

Hyperbola $e^2 = 1 + \frac{b^2}{a^2}$								
Equation	Centre	Focii	Directricies	Major axis	Minor axis	Latus rectum	Verticies	Property
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	(0, 0)	$(\pm\sqrt{a^2+b^2},0)$	$(\sqrt{a^2+b^2})y = \pm a^2$	y = 0	x = 0	$2\frac{b^2}{a}$	(± a, 0) (0, ± b)	$\|SP-S^1P\| = 2a$

The equation of the tangent at (x_1, y_1) , the equation of the chord of contact of (x_1,y_1) and polar of (x_1, y_1) with respect to $S = 0$ is $S_1 = 0$.

Equation of the tangent, chord of contact, and the polar at (x_1, y_1)		
Curve	**Equation**	**$S_1 = 0$**
Parabola	$y^2 = 4ax$	$yy_1 = 2ax + 2ax_1$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

Parametric equations :

Curve	Equation	Parametric point	Parametric equation
Parabola	$y^2 = 4ax$	$t = (at^2, 2at)$	$x = at^2, y = 2at$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\theta = (a\cos\theta, b\sin\theta)$	$x = a\cos\theta, y = b\sin\theta$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\theta = (a\sec\theta, b\tan\theta)$	$x = a\sec\theta, y = b\tan\theta$

Equation of the chord joining two parametric points :

Curve	Equation	Point	Equation of the chord
Parabola	$y^2 = 4ax$	t_1, t_2	$(t_1 + t_2)y - 2x = 2at_1t_2$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	α, β	$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	α, β	$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$

Equation of the tangent at the parametric point :

Curve	Equation	Point	Equation of the tangent	Slope
Parabola	$y^2 = 4ax$	t	$y = \frac{x}{t} + at$	$\frac{1}{t}$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	θ	$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$	$\frac{-b\sin\theta}{a\sin\theta}$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	θ	$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$	$\frac{b}{a\sin\theta}$

Equation of the normal at the parametric point :

Curve	Equation	Point	Equation of the Normal	Slope
Parabola	$y^2 = 4ax$	t	$Y+ \ tx = 2at + at+3$	$-t$
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	θ	$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$	$\frac{a\sin\theta}{b\cos\theta}$
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	θ	$\frac{ax}{\cos\theta}+\frac{by}{\sin\theta}=a^2+b^2$	$\frac{-a\sin\theta}{b}$

Condition for tangency and the point of contact (y = mx + c) :

Curve	Equation	Condition for tangency	Point of contact
Parabola	$y^2 = 4ax$	$c = \frac{a}{m}$	$\left(\frac{c}{m},\frac{2a}{m}\right)$ or $\left(\frac{a}{m^2},\frac{2a}{m}\right)$
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$c^2 = a^2 m^2 + b^2$	$\left(\frac{-a^2m}{c},\frac{b^2}{c}\right)$
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$c^2 = a^2m^2 - b^2$	$\left(\frac{-a^2m}{c},\frac{-b^2}{c}\right)$

Condition that he line lx + my + n = 0 is a tangent :		
Curve	**Equation**	**Condition of tangency**
Parabola	$y^2 = 4ax$	$ln = am^2$
Parabola	$x^2 = 4ay$	$mn = al^2$
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$a^2 l^2 + b^2m^2 = n^2$
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$a^2l^2 - b^2m^2 = n^2$

Equation of the tangent is of form :		
Curve	**Equation**	**Equation of the tangent**
Parabola	$y^2 = 4ax$	$Y=mx + \frac{a}{m}$
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$y = mx \pm \sqrt{a^2m^2+b^2}$
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$y = mx \pm \sqrt{a^2m^2-b^2}$

Equation from which the slopes of the tangents through (x_1,y_1) are given		
Curve	**Equation**	**Condition of tangency**
Parabola	$y^2 = 4ax$	$m^2x_1 - my_1 + a = 0$
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$m^2(x_1^2 - a^2) - 2x_1y_1m + y_1^2 - b^2 = 0$
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$m^2(x_1^2 - a^2) - 2x_1y_1m + y_1^2 + b^2 = 0$

The equation of the chord having the mid point (x_1, y_1) is		
Curve	**Equation**	**Equation of the chord**
Parabola	$y^2 = 4ax$	$yy_1 - 2ax_1 = y_1^2 - 2ax_1$
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}$
The condition that the lines $l_1x + m_1y + n_1 = 0$ and $l_2x + x_2y + n_2 = 0$ may be conjugate with respect to		
Curve	**Equation**	**Condition**
Parabola	$y^2 = 4ax$	$n_1l_2 + n_2l_1 = 2am_1m_2$
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$a^2 l_1 l_2 + b^2m_1m_2 = n_1n_2$
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$a^2 l_1 l_2 - b^2m_1m_2 = n_1n_2$

The locus of the point of intersection of the perpendicular tangents is		
Curve	**Equation**	**Equation of the locus**
Parabola	$y^2 = 4ax$	Directrix – $x + a = 0$
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	Director circle – $x^2+y^2 = a^2 + b^2$
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	Director circle – $x^2+y^2 = a^2 - b^2$

The locus of the points whose chords of contact subtend a right angle at the origin is		
Curve	**Equation**	**Equation of the locus**
Parabola	$y^2 = 4ax$	$x + 4a = 0$
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}-\frac{1}{b^2}$

Ends of latus rectum :		
Curve	**Equation**	**Equation of the locus**
Parabola	$y^2 = 4ax$	(a, 2a), (a, – 2a)
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\left(\pm ae, \pm\frac{b^2}{a}\right)$
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\left(\pm ae, \pm\frac{b^2}{a}\right)$

For any conic, the tangents at the end of Latus rectum, the corresponding directrix and the axis are concurrent.				
Curve	**Equation**	**L**	**L^1**	**Point of concurrency**
Parabola	$y^2 = 4ax$	(a, 2a)	(a, –2a)	z = (– a, 0)
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\left(ae, \frac{b^2}{a}\right)$	$\left(ae, \frac{-b^2}{a}\right)$	$z = \left(\frac{a}{e}, 0\right)$
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\left(ae, \frac{b^2}{a}\right)$	$\left(ae, \frac{-b^2}{a}\right)$	$z = \left(\frac{a}{e}, 0\right)$